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The Hidden Measurement Formalism: What Can Be Explained and Where Quantum Paradoxes Remain

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In the hidden measurement formalism that we have developed in Brussels we explain quantum structure as due to the presence of two effects; (a) a real change of state of the system under influence of the measurement and (b) a lack of knowledge about a deeper deterministic reality of the measurement process. We show that the presence of these two effects leads to the major part of the quantum mechanical structure of a theory describing a physical system, where the measurements to test the properties of this physical system contain the two mentioned effects. We present a quantum machine, with which we can illustrate in a simple way how the quantum structure arises as a consequence of the two effects. We introduce a parameter ε that measures the amount of lack of knowledge on the measurement process, and by varying this parameter, we describe a continuous evolution from a quantum structure (maximal lack of knowledge) to a classical structure (zero lack of knowledge). We show that for intermediate values of ε we find a new type of structure that is neither quantum nor classical. We analyze the quantum paradoxes in the light of these findings and show that they can be divided into two groups: (1) The group (measurement problem and Schrödinger cat paradox) where the paradoxical aspects arise mainly from the application of standard quantum theory as a general theory (e.g., also describing the measurement apparatus). This type of paradox disappears in the hidden measurement formalism. (2) A second group collecting the paradoxes connected to the effect of nonlocality (the Einstein–Podolsky–Rosen paradox and the violation of Bell's inequalities). We show that these paradoxes are internally resolved because the effect of nonlocality turns out to be a fundamental property of the hidden-measurement formalism itself.

1. INTRODUCTION

Quantum mechanics was originally introduced as a noncommutative matrix calculus of observables by Werner Heisenberg (1925) and in parallel

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as a wave mechanics by Erwin Schrödinger (1926). Structurally very different theories, both matrix mechanics and wave mechanics could explain fruitfully the early observed quantum phenomena. Already in the same year the two theories were shown to be realizations of the same, more abstract, ket–bra formalism by Dirac (1958). Some years later, in 1934, John Von Neumann put forward a rigorous mathematical framework for quantum theory in an infinite-dimensional separable complex Hilbert space (Von Neumann, 1955). Matrix mechanics and wave mechanics appear as concrete realizations: the first one if the Hilbert space is l^2, the collection of all square-summable complex numbers, and the second one if the Hilbert space is L^2, the collection of all square-integrable complex functions. The formulation of quantum mechanics in the abstract framework of a complex Hilbert space is now usually referred to as standard quantum mechanics.'

The basic concepts—the vectors of the Hilbert space representing the states of the system and the self-adjoint operators representing the observables—in this standard quantum mechanics are abstract mathematical concepts defined mathematically in an abstract mathematical space, and this is a problem for physicists working to understand quantum mechanics. Several approaches have generalized the standard theory starting from more physically defined basic concepts. John von Neumann and Garett Birkhoff initiated one of these approaches (Birkhoff and von Neumann, 1936) when they analyzed the difference between quantum and classical theories by studying experimental propositions.' They showed that for a given physical system, classical theories have a Boolean lattice of experimental propositions, while for quantum theory the lattice of experimental propositions is not Boolean. Similar fundamental structural differences between the two theories have been investigated by concentrating on different basic concepts. The collection of observables of a classical theory was shown to be a commutative algebra, while this is not the case for the collection of quantum observables (Segal, 1947; Emch, 1984). Luigi Accardi and Itamar Pitovski obtained an analogous result by concentrating on the probability models connected to the two theories: classical theories have a Kolmogorovian probability model, while the probability model of a quantum theory is non-Kolmogorovian (Accardi, 1982; Pitovski, 1989). The fundamental structural differences between the two types of theories, quantum and classical, in different categories was interpreted as indicating also a fundamental difference on the level of the nature of the reality that both theories describe: the micro world should be 'very different' from the macro world. The author admits that he was himself very much convinced of this state of affairs also because very concrete attempts to understand quantum mechanics in a classical way had failed as well: e.g., the many 'physical' hidden variable theories that had been tried out (Selleri, 1990). In this paper we want to concentrate on this problem: how does the quantum world differ from the classical world? We shall do this in the light of the approach that we have been elaborating in Brussels and that we have called the 'hidden measurement formalism.' We concentrate also on the different paradoxes in quantum mechanics: the measurement problem, the Schrödinger cat paradox, the classical limit, the Einstein–Podolsky–Rosen paradox, and the problem of nonlocality. We investigate which of these quantum problems are due to shortcomings of the standard formalism and which point out real physical differences between the quantum and classical worlds.

2. THE TWO MAJOR QUANTUM ASPECTS IN NATURE

As mentioned in the foregoing section, the structural difference between quantum theories and classical theories (Boolean lattice versus non-Boolean lattice of propositions, commutative algebra versus noncommutative algebra of observables, and Kolmogorovian versus non-Kolmogorovian probability structure) is one of the most convincing arguments for the belief in a deep difference between the quantum world and the classical world. During all the years that these structural differences have been investigated (mostly mathematically) there has not been much understanding of the physical meaning of these structural differences. In which way are these structural differences linked to some more intuitive but physically better understood differences between quantum theory and classical theory?

Within the hidden measurement approach we have been able to identify the physical aspects that are at the origin of the structural differences between quantum and classical theories. These are two aspects that characterize the nature of the measurements that have to be carried out to test the properties of the system under study. Let us first carefully formulate these two aspects.

We have a quantum-like theory describing a system under investigation if the measurements needed to test the properties of the system are such that:

(1) The measurements are not just observations, but provoke a real change of the state of the system.

(2) There exists a lack of knowledge about the reality of what happens during the measurement process.

It is the lack of knowledge (2) that is theoretically structured in a non-Kolmogorovian probability model. In a certain sense it is possible to interpret the second aspect, the presence of the lack of knowledge on the reality of the measurement process, as the presence of 'hidden measurements' instead of 'hidden variables.' Indeed, if a measurement is performed with such a lack of knowledge, then this is actually the classical mixture of a set of classical hidden measurements, where for such a classical hidden measurement there would be no lack of knowledge. In an analogous way as in a hidden variable theory, the quantum state is a classical mixture of classical states. This is why we have called the formalism that we are elaborating in Brussels and that consists in formalizing in a mathematical theory the physical situations containing the two mentioned aspects, the 'hidden measurement formalism'.

3. THE QUANTUM MACHINE PRODUCING QUANTUM STRUCTURE

After we had identified the two aspects (1) and (2) it was not difficult to invent a quantum machine fabricated only with macroscopic materials and producing a quantum structure isomorphic to the structure of a two-dimensional complex Hilbert space, describing for example, the spin of a quantum particle with spin 1/2 (Aerts, 1985, 1986, 1987). This quantum machine has been presented on different occasions (Aerts, 1988a,b, 1991a, 1995) and therefore we shall only introduce it briefly here for the sake of completeness.

The machine that we consider consists of a physical entity S that is a point particle P that can move on the surface of a sphere, denoted *surf*, with center O and radius 1. The unit vector v where the particle is located on *surf* represents the state p_v of the particle (see Fig. 1a). For each point $u \in surf$, we introduce the following measurement e_u. We consider the diametrically opposite point $-u$, and install a piece of elastic of length 2 such that it is fixed with one of its endpoints in u and the other endpoint in $-u$. Once the elastic is installed, the particle P falls from its original place v orthogonally onto the elastic and sticks on it (Fig. 1b). Then the elastic breaks and the particle P, attached to one of the two pieces of the elastic (Fig. 1c), moves to one of the two endpoints u or $-u$ (Fig. 1d). Depending on whether the particle P arrives at u (as in Fig. 1) or at $-u$, we give the outcome o_1^u or o_2^u to e_u. We can easily calculate the probabilities corresponding to the two possible outcomes.

Therefore we remark that the particle P arrives at u when the elastic breaks at a point in the interval L_1 (which is the length of the piece of the elastic between $-u$ and the point where the particle has arrived, or $1 + \cos\theta$), and arrives at $-u$ when it breaks at a point in the interval L_2 ($L_2 = L - L_1 = 2 - L_1$). We make the hypothesis that the elastic breaks uniformly, which means that the probability that the particle, being in state p_v, arrives at u is given by the length of L_1 divided by the length of the total elastic (which is 2). The probability that the particle in state p_v arrives at $-u$ is the length of L_2 (which is $1 - \cos\theta$) divided by the length of the total elastic.

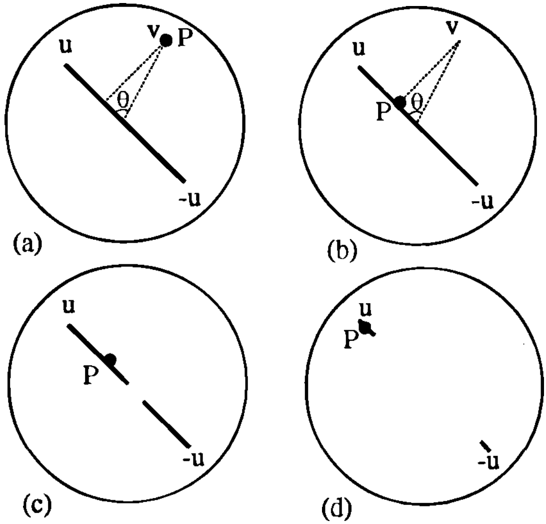

Fig. 1. A representation of the quantum machine. (a) The physical entity P is in state p_v at the point v, and the elastic corresponding to the measurement e_u, is installed between the two diametrically opposed points u and $-u$. (b) The particle P falls orthogonally onto the elastic and sticks to it. (c) The elastic breaks and the particle P is pulled toward the point u, such that (d) it arrives at the point u, and the measurement e_u gets the outcome o_1^u.

If we denote these probabilities respectively by $P(o_1^u, p_v)$ and $P\ (o_2^u, p_v)$ we have

$$P(o_1^u, p_v) = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2}, \qquad P(o_2^u, p_v) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{1}$$

These transition probabilities are the same as the ones related to the outcomes of a Stern–Gerlach spin measurement on a spin-1/2 quantum particle, of which the quantum spin state in direction $v = (\cos\theta \sin\theta, \sin\theta \sin\theta, cos\ \theta)$, denoted by ψ_v, and the measurement e_u corresponding to the spin measurement in direction $u = (\cos\beta \sin\alpha, \sin\beta \sin\alpha, \cos\alpha)$, is described respectively by the vector and the self-adjoint operator of a two-dimensional complex Hilbert space,

$$\psi_v = (e^{-i\phi/2}\cos\theta/2, e^{i\phi/2}\sin\theta/2), \qquad H_u = \frac{1}{2}\begin{pmatrix} \cos\alpha & e^{-i\beta}\sin\alpha \\ e^{i\beta}\sin\alpha & \cos\alpha \end{pmatrix} \tag{2}$$

We can easily see now the two aspects in this quantum machine that we have identified in the hidden measurement approach to give rise to the quantum structure. The state of the particle P is effectively changed by the measuring apparatus (p_v changes to p_u or to p_{-u} under the influence of the measuring process), which identifies the first aspect, and there is a lack of knowledge on the interaction between the measuring apparatus and the particle, namely the lack of knowledge of where exactly the elastic will break, which identifies the second aspect. We can also easily understand now what is meant by the term 'hidden measurements.' Each time the elastic breaks at one specific point λ, we could identify the measurement process that is carried out afterward as a hidden measurement e_u^λ. The measurement e_u is then a classical mixture of the collection of all measurement e_u^λ: namely e_u consists in choosing at random one of the e_u^λ and performing this chosen e_u.

4. THE QUANTUM CLASSICAL RELATION

First we remark that our group in Brussels has shown that such a hidden measurement model can be built for any arbitrary quantum entity (Aerts, 1985, 1986, 1987; Coecke, 1995a,b). However, the hidden measurement formalism is more general than standard quantum theory. Indeed, it is very easy to produce quantum-like structures that cannot be represented in a complex Hilbert space (Aerts, 1986).

If the quantum structure can be explained by the presence of a lack of knowledge on the measurement process, we can go a step further, and wonder what types of structure arise when we consider the original models, with a lack of knowledge on the measurement process, and introduce a variation of the magnitude of this lack of knowledge. We have studied the quantum machine under varying 'lack of knowledge', parametrizing this variation by a number $\varepsilon \in [0,1]$, such that $\varepsilon = 1$ corresponds to the situation of maximal lack of knowledge, giving rise to a quantum structure, and $\varepsilon = 0$ corresponds to the situation of zero lack of knowledge, generating a classical structure. Other values of ε correspond to intermediate situations and give rise to a structure that is neither quantum nor classical (Aerts *et al.*, 1992, 1993; Aerts and Durt, 1994a,b). We have called this model the ε-model and want to introduce it again in this paper to explain how some of the quantum paradoxes are solved in the hidden measurement formalism.

4.1. The ϵ-Model

We start from the quantum machine, but now introduce different types of elastic. An ε,d-elastic consists of three different parts: a lower part where it is unbreakable, a middle part where it breaks uniformly, and an upper part where it is again unbreakable. By means of the two parameters $\varepsilon \in [0,1]$ and $d \in [-1 + \varepsilon, 1 - \varepsilon]$, we fix the sizes of the three parts in the following way. Suppose that we have installed the ε,d-elastic between the points $-u$ and u of the sphere. Then the elastic is unbreakable in the lower part from

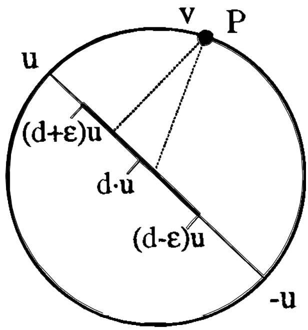

Fig. 2. A representation of the measurement $e^{\varepsilon}_{u,d}$. The elastic breaks uniformly between the points $(d - \varepsilon)\ u$ and $(d + \varepsilon)u$, and is unbreakable at other points.

$-u$ to $(d - \varepsilon) \cdot u$, it breaks uniformly in the part from $(d - \varepsilon) \cdot u$ to $(d + \varepsilon) \cdot u$, and it is again unbreakable in the upper part from $(d + \varepsilon) \cdot u$ to u (see Fig. 2).

An e_u measurement performed by means of an ε,d-elastic shall be denoted by e^{ε}_{u},d. We have the following cases:

(1) $v \cdot u \leq d - \varepsilon$. The particle sticks to the lower part of the ε,d-elastic, and any breaking of the elastic pulls it down to the point $-u$. We have $P^{\varepsilon}(o^u_1, p_v) = 0$ and $P^{\varepsilon}(o^u_2, p_v) = 1$.

(2) $d - \varepsilon < v \cdot u < d + \varepsilon$. The particle falls onto the breakable part of the ε,d-elastic. We can easily calculate the transition probabilities and find

$$P^{\varepsilon}(o^u_1, p_v) = \frac{1}{2\varepsilon}(v \cdot u - d + \varepsilon), \qquad P^{\varepsilon}(d^u_2, p_v) = \frac{1}{2\varepsilon}(d + \varepsilon - v \cdot u) \quad (3)$$

(3) $d + \varepsilon \leq v \cdot u$. The particle falls onto the upper part of the ε,d-elastic, and any breaking of the elastic pulls it upward, such that it arrives at u. We have $P^{\varepsilon}\ (o^u_1, p_v) = 1$ and $P^{\varepsilon}(o^u_2, p_v) = 0$.

We are now in a very interesting situation from the point of view of the structural studies of quantum mechanics. Since the ε-model describes a continuous transition from quantum to classical, its mathematical structure should be able to teach us *what the structural shortcomings are of the standard Hilbert space quantum mechanics.* Therefore we have studied the ε-model in existing mathematical approaches that are more general than standard quantum mechanics: the lattice approach, the probabilistic approach, and the *-algebra approach.

4.2. The Lattice Approach

In this lattice approach there is a well-known axiomatic scheme that reduces the approach to standard quantum mechanics if certain axioms are fulfilled. For intermediate values of ε, that is, $0 < \varepsilon < 1$, we find that two of the five axioms needed in the lattice approach to reconstruct standard quantum mechanics are violated. The axioms that are violated are the weak-modularity and the covering law, and it is precisely those axioms that are needed to recover the vector space structure of the state space in quantum mechanics (Aerts *et al.*, 1993; Aerts and Durt, 1994a, b).

4.3. The Probabilistic Approach

If we take the case of vanishing fluctuations ($\varepsilon = 0$), do we obtain the Kolmogorovian theory of probability? This would be most interesting, since then we would have constructed a macroscopic model with an understandable structure (i.e., we can see how the probabilities arise) and a quantum and a classical behavior. We proposed to test the polytopes for a family of conditional probabilities (D. Aerts, 1995). The calculations can be found in S. Aerts (1996) and the result was what we hoped for: a macroscopic model with a quantum and a Kolmogorovian limit. For intermediate values of the fluctuations ($0 < \varepsilon < 1$) the resulting probability model is neither quantum nor Kolmogorovian: we have identified here a new type of probability model that is quantum-like, but not really isomorphic to the probability model found in a complex Hilbert space.

4.4. The *-Algebra Approach

The *-algebra provides a natural mathematical language for quantum mechanical operators. We applied the concepts of this approach to the ε-model to find that an operator corresponding to an ε measurement is linear if and only if $\varepsilon = 1$ (Aerts and D'Hooghe, 1996). This means that for the classical and intermediate situations the observables cannot be described by linear operators.

Conclusion. Quantum theory and classical theories appear as special cases ($\varepsilon = 1$, $\varepsilon = 0$), and the general intermediate case, although quantum-like, cannot be described in standard Hilbert space quantum mechanics.

4.5. The Measurement Problem and Schrödinger Cat Paradox

The result stated in the Conclusion means that all the paradoxes of standard quantum theory that are due to the fact that quantum theory is used as a universal theory also being applied to a macroscopic system, for example, the measuring apparatus, are not present in our hidden measurement formalism. We explicitly have in mind the 'measurement problem' and the 'Schrödinger cat paradox'. Indeed, the measurement apparatus should be described by a classical model in our approach, and the physical system eventually by a quantum model. The problem of the presence of quantum correlations between physical system and measuring apparatus, as it presents itself in the standard theory, takes a completely different aspect. We are working now on the elaboration of a concrete description of the measurement process within the hidden measurement formalism (Aerts and Durt, 1994b). Our result also shows that it is possible in the hidden measurement formalism to formulate a 'classical limit', namely as a continuous transition from quantum to classical (Aerts *et al.*, 1993).

5. NONLOCALITY AS A GENUINE PROPERTY OF NATURE

The measurement problem and paradoxes equivalent to Schrödinger's cat paradox disappear in the hidden measurement formalism because standard quantum mechanics appears only as a special case, the situation of maximum fluctuations. All quantum mysteries connected to the effect of 'nonlocality' remain (Einstein–Podolsky–Rosen paradox and the violation of the Bell inequalities). It is even so that nonlocality unfolds itself as a fundamental aspect of the hidden measurement approach. This is due to the fact that if we explain the quantum structures as is done in the hidden measurement formalism a quantum measurement has two concrete physical effects: (1) it changes the state of the system and (2) it produces probability due to a lack of knowledge about the nature of this change. With the quantum machine we have given a macroscopic model for a spin measurement of a spin-1/2 particle. If we apply the hidden measurement formalism to the situation of a quantum system described by a wave function $\psi(x)$ and to a position or a momentum measurement performed on this system, we also have the two mentioned effects. For example, in the case of a position measurement, the detection apparatus changes the state of the quantum system, in the sense that it 'localizes' the quantum system in a specific place of space, and the probabilities that are connected with this measurement are due to the fact that we lack knowledge about the specific way in which this localization takes place. This means that 'before the detection has taken place' the quantum system was in general not localized: it was not present in a specific region of space. In other words, quantum systems are fundamentally nonlocal systems: the wave function $\psi(x)$ describing such a quantum state is not interpreted as a wave that is present in space, but as indicating those regions of space were the particle can be localized, where $\int_R |\psi(x)|^2 \, dx$ is the probability that this localization will happen in region R. A similar interpretation must be given to the momentum measurement of a quantum entity: the quantum entity has no momentum before the measurement, but the measurement creates partly this momentum.

Aerts *et al.* (1993) calculated the ε-situation for a quantum system described by a wave function $\psi(x)$, element of the Hilbert space of all square-integrable complex functions, and we found the following very simple procedure. Suppose that ε is given, and the state of the quantum system is described by the wave function $\psi(x)$, and $\phi(x)$ is the corresponding probability distribution [hence $\phi(x) = |\psi(x)|^2$]. We cut, by means of a constant function ϕ_Ω, a piece of the function $\phi(x)$ such that the surface contained in the cutoff piece equals ε (see step 1 of Fig. 3). We move this piece of function to the x axis (see step 2 of Fig. 3). Then we renormalize by dividing by ε (see step 3 of Fig. 3). If we proceed in this way for smaller values of ε, we finally arrive at a delta function for the classical limit $\varepsilon \to 0$, and the delta function is located at the original maximum of the quantum probability distribution. We point out that the state $\psi(x)$ of the physical system is not changed by this ε-procedure, it remains always the same state, representing the same physical reality. It is the regime of lack of knowledge, together with the detection measurement, that changes with varying ε. For $\varepsilon = 1$ this regime is one of maximum lack of knowledge on the process of localization, and this lack of knowledge is characterized by the spread of the probability distribution $\phi(x)$. For an intermediate value of ε, between 1 and 0, the spread of the probability

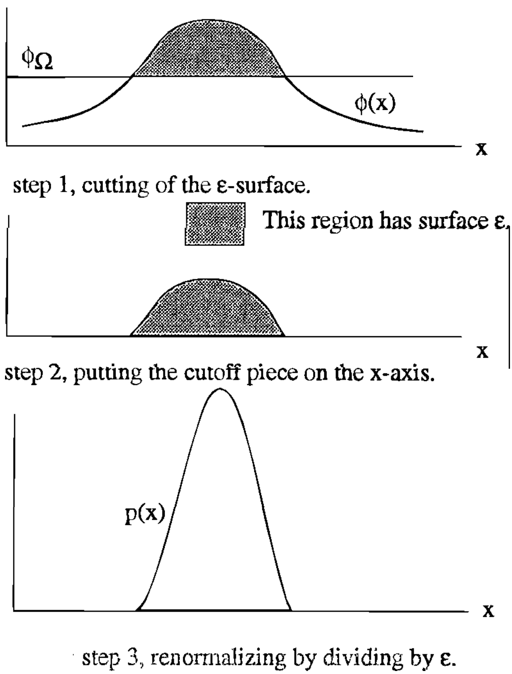

Fig. 3. The classical limit procedure.

distribution has decreased (see Fig. 3) and for zero fluctuations the spread is 0. Let us also try to see what becomes of the nonlocal behavior of quantum entities taking into account the classical limit procedure that we propose. Suppose that we consider a double-slit experiment, then the state p of a quantum entity having passed the slits can be represented by a probability function $p(x)$ of the form represented in Fig. 4. We can see that the nonlocality presented by this probability function gradually disappears when ε becomes smaller, and, in the case where $p(x)$ has only one maximum, finally disappears completely. When there are no fluctuations on the measuring apparatus used to detect the particle, it shall be detected with certainty in one of the slits, and always in the same one. If $p(x)$ has two maxima (one behind slit 1, and the other behind slit 2) that are equal, the nonlocality does not disappear. Indeed, in this case the limit function is the sum of two delta functions (one behind slit 1 and one behind slit 2). So in this case the nonlocality remains present even in the classical limit. If our procedure for the classical limit is a correct one, also macroscopic classical entities can be in nonlocal states. How does it come about that we do not find any sign of this nonlocality in the classical macroscopic world? This is due to the fact that the set of states representing a situation where the probability function has more than one maximum has measure zero, compared to the set of all possible states, and moreover these states are ‘unstable’. The slightest perturbation will destroy the symmetry of the different maxima, and hence shall give rise to one point

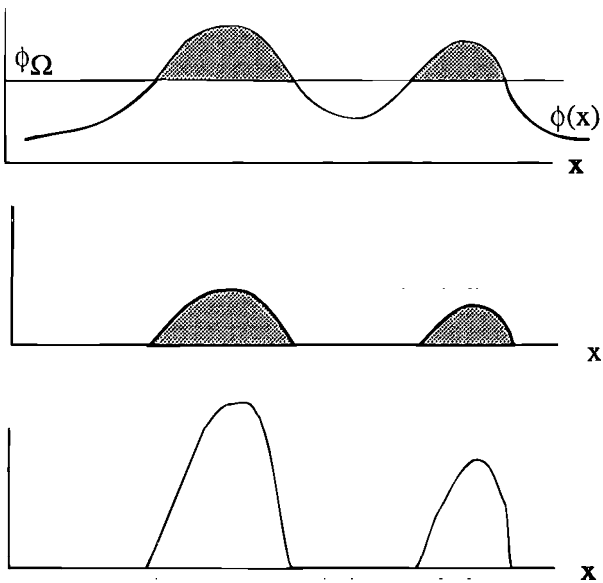

Fig. 4. The classical limit procedure in the situation of a nonlocal quantum state.

of localization in the classical limit. Also, classical macroscopic reality is nonlocal, but the local model that we use to describe it gives the same statistical results, and hence cannot be distinguished from the nonlocal model.

6. THE VIOLATION OF BELL INEQUALITIES: QUANTUM, CLASSICAL, AND INTERMEDIATE

It is interesting to consider the violation of Bell's inequalities within the hidden measurement formalism. The quantum machine, as presented in Section 3, delivers us a macroscopic model for the spin of a spin-1/2 quantum entity, and starting with this model it is possible to construct a macroscopic situation, using two of these models coupled by a rigid rod, that represents faithfully the situations of two entangled quantum particles of spin-1/2 (Aerts, 1991b). The 'nonlocal' element is introduced explicitly by means of a rod that connects the two sphere models. We also have studied this EPR situation of entangled quantum systems by introducing the ε variation of the amount of lack of knowledge on the measurement processes and could show that one violates the Bell inequalities even more for classical but nonlocally connected systems, that is, $\varepsilon = 0$. This illustrates that the violation of the Bell inequalities is due to the nonlocality rather then to the indeterministic character of quantum theory, and that the quantum indeterminism (for values of ε greater than 0) tempers the violation of the Bell inequalities (Aerts *et al.*, 1996a). This idea has been used to construct a general representation of entangled states (hidden correlations) within the hidden measurement formalism (Coecke, 1996a,b; Coecke *et al.*, 1996).

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